Fluctuations and Pattern Selection near an Eckhaus Instability

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We study the effect of fluctuations in the vicinity of an Eckhaus instability. Fluctuations smear out the stability limit into a region in which fluctuations and nonlinearities dominate the decay of unstable states. We also find an effective stability boundary that depends on the intensity of fluctuations. A numerical solution of the stochastic Swift-Hohenberg equation in one dimension is used to test these predictions and to study pattern selection when the initial unstable state lies within the fluctuation dominated region. The nonlinear relaxation is shown to exhibit a scaling form.

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Recent experimental studies of Rayleigh-Bénard convection in CO_2 gas [1], convection in binary mixtures [2], electrohydrodynamic convection in nematic liquid crystals [3], and Taylor-Couette flow [4] have detected random fluctuations of thermal origin that are strong enough to be analyzed quantitatively. Whereas the effect of thermal fluctuations on thermodynamic equilibrium is now well understood in general terms, much less is known about their effect in the vicinity of nonequilibrium instabilities. Pioneering work in this direction was done by Graham [5], and Swift and Hohenberg [6]. It was believed, however, that since characteristic thermal energies are several orders of magnitude smaller than the characteristic driving or dissipative energies involved in many nonequilibrium instabilities, the effect of fluctuations would be far too small to be observable. The results of these recent experiments have renewed interest in the subject since they have opened the possibility of finding nontrivial phenomena due to fluctuations close to nonequilibrium instability points.

We study in this paper the effect of fluctuations on the Eckhaus instability [7]. The Eckhaus instability is a longitudinal instability often exhibited in systems that display patterns which are spatially periodic. It has been exhaustively studied in many systems, including, for example, the stability of a set of parallel convective rolls in Rayleigh-Bénard convection [8], the stability of a periodic array of cells during directional solidification [9], and instabilities of stationary standing Faraday waves [10]. We also note particularly detailed experimental studies of the Eckhaus instability in electrohydrodynamic convection in nematic liquid crystals by Lowe and Gollub [11] and Rasenat, Braun, and Steinberg [12]. Three main issues are addressed here: the fluctuation-induced smearing of the classical Eckhaus boundary, the nonlinear relaxation of the unstable solution, and the asymptotic periodicity of the new stable solution following the instability (this is often referred to as pattern selection).

In the absence of fluctuations, the Eckhaus boundary of potential or gradient systems is located at the line where both the first and second functional derivatives of the appropriate Lyapunov functional vanish; hence it separates regions of metastability and instability. This is the analog of a spinodal line in a first order phase transition. Our study can be motivated by analogy with this latter case: If fluctuations are included in the description of a first order phase transition, the spinodal line ceases to exist [13]. Instead, states that are not stable are classified as metastable or unstable with reference to a particular temporal scale of evolution (observation time). States that do not decay within such a scale are said to be metastable, and unstable otherwise. The often narrow region that separates both types of behavior is shifted with respect to the spinodal line, and defines an effective stability boundary known as the "cloud point." We found here related phenomena, namely, the smearing of the Eckhaus boundary for finite amplitudes of the fluctuations and the emergence of a different time scale for the decay of the unstable state. This scale depends on the intensity of the fluctuations and identifies the observable stability boundary. Our analysis of the fluctuation-dominated regime is similar in spirit to Binder's determination of the range of validity of a linearized description of phase separation [14]. He derived the analog of a Ginzburg criterion to describe the effect of fluctuations on the long wavelength instability known as spinodal decomposition. Finally, we also discuss pattern selection when the initial state is within the transition region.

Although we expect our analysis to be of wider applicability, we restrict ourselves here to the one-dimensional stochastic Swift-Hohenberg (SSH) equation [6],

$$\frac{\partial \psi(x,t)}{\partial t} = \left[\epsilon - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right] \psi(x,t) - \psi(x,t)^3 + \sqrt{D}\xi(x,t) , \tag{1}$$

which is known to be a model of Rayleigh-Benard convection near onset. The periodicity of the stationary so-

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lutions of Eq. (1) is commensurate with the convective rolls, ϵ is the reduced Rayleigh number, and $\xi(x,t)$ is a Gaussian random process satisfying

$$\langle \xi(x,t) \rangle = 0, \quad \langle \xi(x,t)\xi(x',t') \rangle = 2\delta(x-x')\delta(t-t'). \tag{2}$$

The value of D is proportional to the intensity of the thermal fluctuations in the fluid. For $D=0$ and $\epsilon > 0$, Eq. (1) admits periodic stationary solutions $\psi_q(x)$ of wave number q with $q_{-L} < q < q_L$, and $q_{\pm L} = \sqrt{1 \pm \sqrt{\epsilon}}$ [15]. However, only those solutions with $q_{-E} < q < q_E$ ($q_{-L} < q_{-E} < q_E < q_L$) are linearly stable. Periodic solutions with q outside this range exhibit a modulational instability known as the Eckhaus instability.

The decay of the unstable solution is triggered by fluctuations. In order to study such decay, we linearize Eq. (1) around a stationary periodic solution $\psi_{q_i}(x)$, with $q_E < q_i < q_L$. Linearization leads to an eigenvalue problem for a spatially periodic operator. According to Floquet's theorem, the normalized eigenfunctions $f_{q_i,k}(x)$ are of the form [16]

$$f_{q_i,k}(x) = \sum_{n=-\infty}^{\infty} \omega^n_{q_i,k} e^{i(nq_i+k)x} , \tag{3}$$

each associated with an eigenvalue $\lambda(q_i,k)$, which determines the growth rate of that particular eigenfunction. Because the SH equation is real, these eigenvalues satisfy the symmetry relation $\lambda(q_i,-k) = \lambda(q_i,k)$. Also, the invariance of the SH equation under spatial translations implies $\lambda(q_i, k=0) = 0$. Let $\pm k_m$ be the values of k that maximize $\lambda(q_i,k)$ and let $f_m(x)$ be the real (and normalized) combination of their associated eigenfunctions. Then, it is possible to describe the early stages of the decay of the unstable state by a linear stochastic equation governing the amplitude $u(t)$ of $f_m(x)$, in the eigenfunction expansion of $\psi(x,t) - \psi_{q_i}(x)$:

$$\dot{u}(t) = \lambda_m u(t) + \sqrt{D}\eta(t) . \tag{4}$$

$\eta(t)$ is a white Gaussian random process which results from the projection of $\xi(x,t)$ onto $f_m(x)$: $\eta(t) = \int dx\, f_m(x)\xi(x,t)$, the average of which is zero. When the eigenfunctions in Eq. (3) are normalized, the variance of η turns out to be independent of the initial periodicity q_i. Equation (4) is valid until a time t_p at which the amplitude u becomes large enough such that nonlinearities become important. The solution of Eq. (4) with initial condition $u(t=0)=0$ is $u(t) = h(t)e^{\lambda_m t}$, with

$$h(t) = \sqrt{D} \int_0^t ds\, e^{-\lambda_m s}\eta(s) . \tag{5}$$

The amplitude $h(t)$ is a Gaussian random process, averaging to zero, and with a time-dependent variance. From its definition it is easy to see that it becomes stationary after a time $t_m \sim \lambda_m^{-1}$. After this time, one can replace $h(t)$ by its long time limit, $h(t=\infty)$, which is a Gaussian variable of standard deviation $\sqrt{\langle h^2 \rangle} = \sqrt{D/\lambda_m}$. Within this approximation, the calculation of the average time t_p at which $u(t)$ crosses a given reference value u_T is standard and gives [17]

$$t_p = \frac{1}{\lambda_m} \ln \left(u_T \sqrt{\frac{\lambda_m}{D}} \right) . \tag{6}$$

Two characteristic time scales emerge from this analysis. The time t_m signals the end of the fluctuation dominated regime and the beginning of a linear deterministic regime. By choosing u_T equal to a given (small) fraction of the final saturation value of u, t_p can be interpreted as the time at which nonlinear terms begin to be important. The approximation of replacing $h(t)$ by its asymptotic value is obviously true only if $t_m \ll t_p$, which implies $u_T^2 \lambda_m \gg D$. When

$$\lambda_m \approx \frac{D}{u_T^2} \tag{7}$$

there is no clear separation of time scales, and there will not be a distinct linear deterministic regime. Nonlinear effects then become important even in the early fluctuation dominated regime.

The arguments given above are of a general nature. Consider now an initially periodic state of wave number $q_i \gtrsim q_E$. The linear growth rate of the most unstable eigenfunction vanishes near the Eckhaus boundary as

$$\lambda_m \sim (q_i - q_E)^2. \tag{8}$$

This can be seen by noting that the symmetry properties of $\lambda(q_i,k)$ stated above imply that, for small k, $\lambda(q_i,k) \approx ak^2 - bk^4 + \cdots$, where a and b are functions of q_i and ϵ. b must be positive to avoid instabilities with arbitrarily large wave number, and a changes sign at the Eckhaus boundary $q_i = q_E$. Then, to first order, $a \sim q_i - q_E$. The maximum $\lambda_m(q_i)$ of $\lambda(q_i,k)$ is at $k_m = \pm\sqrt{a/2b}$, which leads to Eq. (8). In addition, it can be seen that the most important wave number in the Fourier expansion of the eigenfunction $f_m(x)$ is $q_m \equiv q_i - k_m$. From the expression for k_m we get

$$q_i - q_m \sim (q_i - q_E)^{\frac{1}{2}}. \tag{9}$$

We now define a transition region of initial wave numbers q_i around q_E, within which the extent of the linear deterministic regime vanishes. From Eqs. (7) and (8) we find that this transition region is determined by

$$0 \le q_i - q_E \le \frac{C}{u_T}\sqrt{D}, \tag{10}$$

where C is a constant of order 1. Note that u_T is finite near the Eckhaus boundary, since it has to be taken as a finite fraction of the final saturation amplitude of the fastest growing mode, and this quantity has no singularity across the Eckhaus boundary.

These results are restricted to an initially unstable state; hence they only apply to $q_i > q_E$. Initial states with $q_i < q_E$ which are linearly stable according to a deterministic calculation can also decay because of fluctuations, extending the transition region to $q_i < q_E$. The smearing of the Eckhaus boundary and the concomitant reduction of the region of linearly stable states can be thought of as an effective shift of the Eckhaus boundary to a new value $\tilde{q}_E < q_E$. It should be stressed, however, that this new effective boundary is only defined with reference to a given observation time for the decay of the metastable state.

We next address the implications of our analysis on the issue of pattern selection. Pattern selection refers to the determination of a global wave number q_f of a configuration obtained after a long time, as a function of the initial wave number q_i. The wave number q_f is defined here as $q_f = \pi n_f$, where n_f is the number of nodes of the configuration per unit length [18]. If q_i lies outside the transition region, the initial stages of the decay of the unstable state are well described by linear theory, and the fastest growing mode in the linear regime, q_m, will mostly determine q_f. In this case, the evolution does not lead in observable time scales to a periodic configuration of wave number $q_{\rm min}$ that minimizes the Lyapunov functional associated with Eq. (1). In fact, earlier numerical simulations of the SSH equation showed that $q_f \approx q_m$ when q_i is not too close to the Eckhaus boundary [19]. These results were consistent with previous simulations of the amplitude equation also in the absence of noise [20]. On the other hand, if q_i lies inside the transition region, nonlinearities and fluctuations are likely to alter these conclusions.

To test our predictions we carried out a computer simulation study of the one-dimensional SSH equation. Complete details of the algorithm used can be found elsewhere [21]. We chose $\epsilon = 0.056\,25$ and discretized Eq. (1) on an evenly spaced grid with $N = 8192$ nodes and $\Delta x = 2\pi/32$. The equation was integrated forward in time with $\Delta t = 1 \times 10^{-4}$, and the results averaged over a number of independent runs, typically 10–20. With this choice of Δx, $\Delta q = 2\pi/N\Delta x = 0.004$.

Figure 1 presents the average value of q_f as a function of q_i for $D/\Delta x = 0.05, 0.1$, and 0.2. The dashed line is the value of q_m found by numerically calculating the Floquet spectrum on the same grid used to solve Eq. (1). The analysis also yields $q_E \approx 1.201$. Simulation points for $q_i < q_E$ are also shown in Fig. 1: although in the stable range according to the deterministic calculation they are seen to decay. Furthermore, for q_i far enough from the Eckhaus boundary the value of q_f obtained from the simulation approaches q_m (which in the same limit approaches 1). However as $q_i - q_E$ is decreased, the data deviate from q_m. The size of the region where q_f deviates from q_m identifies a transition region that increases with D, in a manner consistent with the prediction of Eq. (10). By taking the value of $Cu_T^{-1} \approx 0.219$, Eq. (10) gives a size for this transition region extending up to $q_i \approx 1.25$, 1.27, and 1.30, for $D/\Delta x = 0.05$, 0.1, and 0.2, respectively, which is consistent with the trend of the data in Fig. 1. A more detailed comparison of Eq. (10) with our numerical results has not been attempted since Eq. (10) only gives an estimate with undetermined coefficients.

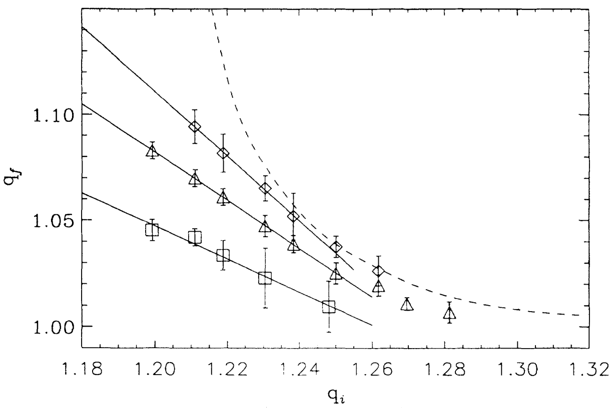

FIG. 1. Final average wave number q_f as a function of the initial wave number q_i, for several values of the intensity of the fluctuations $D/\Delta x$: $\diamond$, 0.05; Δ, 0.1; and $\Box$, 0.2. The straight solid lines are linear fits to the data. The dashed line is q_m, the wave number corresponding to the eigenfunction of fastest growth given by linear theory. The error bars in the figure represent the 2σ confidence interval for the sample studied.

In contrast with the prediction of linear theory [Eq. (9)], the simulation points for low enough q_i lie on straight lines, so that the exponent 1/2 in Eq. (9) is changed by fluctuations to a value close to 1. We have defined an effective stability boundary $\tilde{q}_E$, a function of D, that separates stable from unstable states within our simulation time ($t \approx 100$; see Fig. 1). We define it to be equal to the intersection point between the straight lines in Fig. 1 and the line $q_i = q_f$. This gives $\tilde{q}_E = 1.16$, 1.14, and 1.11 for $D/\Delta x = 0.05, 0.1$, and 0.2, respectively. In addition, our data disagree with standard criteria for pattern selection according to which $q_f = q_m$ or $q_f = q_{\rm min}$, when q_i lies in the transition region.

Last we describe our numerical results for the temporal evolution of the dominant periodicity of the configuration $q(t)$ after it becomes linearly unstable. The function $q(t)$ is defined as $q(t) = \pi n(t)$, where $n(t)$ is the number of nodes of $\psi(x,t)$ per unit length, averaged over independent runs. The decay is found to take place in a characteristic time τ with an apparent divergence as q_i approaches $\tilde{q}_E$. A similar behavior has been observed in simpler zero-dimensional stochastic models used to describe the region that separates a metastable from an unstable state [22]. There, it is possible to characterize the temporal evolution in terms of scaling relations. With this motivation in mind, we have found that our data are well described by a scaling relation of the form

$$q(t) - \tilde{q}_E = (q_i - \tilde{q}_E) f(t(q_i - \tilde{q}_E)^z), \tag{11}$$

as shown in Fig. 2 for $D/\Delta x = 0.1$. The best scaling is found for $z = 1.7$. Scaling is also observed for other values of D, with z in the range 1.65–1.75.

Scaling relations as Eq. (11) could also be worth ex-

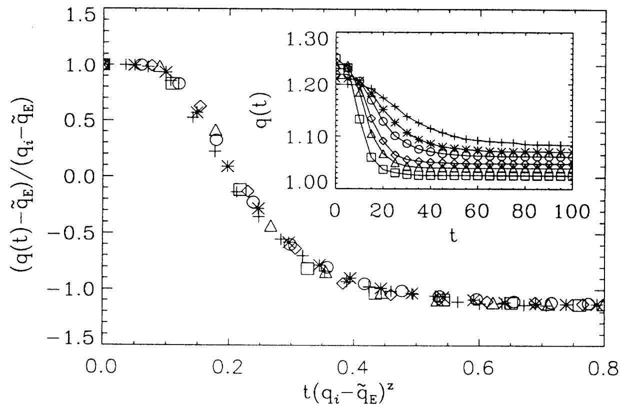

FIG. 2. Scaled average periodicity of the configuration as a function of time after the instability (the unscaled values are shown in the inset) for $D/\Delta x = 0.1$. The value of $z = 1.7$ has been used. The values of q_i shown are (from top to bottom in the right of the inset and the corresponding symbols in the figure) $q_i = 1.119, 1.211, 1.219, 1.230, 1.238$, and 1.250.

ploring to describe the transient dynamics of systems undergoing generalized forms of spinodal decomposition in which a large number of long-lived metastable states could occur.

Detailed comparison of our results with experiments would certainly require at least the consideration of a two-dimensional SSH equation, but we expect that the general ideas exposed here, namely, the existence of a transition region separating stable and unstable states and the identification of an observable stability limit shifted with respect to q_E and related to an increase of the characteristic relaxation times, are of wider applicability. We note, for example, qualitative similarities with some experimental findings. Lowe and Gollub [11] observed the decay of states in the deterministically stable range and their evolution towards states of wave number intermediate between q_m and $q_{\rm min}$. In addition, in the experiments by Rasenat, Braun, and Steinberg [12], the Eckhaus boundary was determined in a way very similar to ours, and it was noted that, in contrast with deterministic theory, the experimental results for $q_f - q_i$ and the characteristic time scales of the decay of the unstable state as a function of q_i were well fitted by straight lines. Further experimental work, especially in systems in which the intensity of the thermal fluctuations is relatively large, is certainly needed to clarify whether the effects described above can be explained by our theoretical arguments.

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[1] E. Bodenschatz, S.W. Morris, J.R. de Bruyn, D.S. Cannell, and G. Ahlers, in *Pattern Formation in Complex Dissipative Systems*, edited by S. Kai (World Scientific, Singapore, 1992).
[2] W. Schöpf and I. Rehberg, Europhys. Lett. **17**, 321 (1992).
[3] I. Rehberg, S. Rasenat, M. de la Torre, W. Schöpf, F. Hörner, G. Ahlers, and H.R. Brand, Phys. Rev. Lett. **67**, 596 (1991).
[4] K.L. Babcock, G. Ahlers, and D.S. Cannel (to be published).
[5] R. Graham, Phys. Rev. A **10**, 1762 (1974).
[6] J. Swift and P.C. Hohenberg, Phys. Rev. A **15**, 319 (1977); P.C. Hohenberg and J. Swift, Phys. Rev. A **46**, 4773 (1992).
[7] W. Eckhaus, *Studies in Non-linear Stability Theory* (Springer, New York, 1965); L. Kramer and W. Zimmermann, Physica (Amsterdam) **16D**, 221 (1985).
[8] P. Manneville, *Dissipative Structures and Weak Turbulence* (Academic, New York, 1990).
[9] K. Brattkus and C. Misbah, Phys. Rev. Lett. **64**, 1935 (1990); A. Ghazali and C. Misbah, Phys. Rev. A **46**, 5026 (1992).
[10] S. Douady, S. Fauve, and O. Thual, Europhys. Lett. **10**, 309 (1989); S.T. Milner, J. Fluid Mech. **225**, 81 (1991).
[11] M. Lowe and J.P. Gollub, Phys. Rev. Lett. **55**, 2575 (1985).
[12] S. Rasenat, E. Braun, and V. Steinberg, Phys. Rev. A **43**, 5728 (1991).
[13] J.D. Gunton, M. San Miguel, and P. Sahni, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, New York, 1983), Vol. 8.
[14] K. Binder, Phys. Rev. A **29**, 341 (1984).
[15] Y. Pomeau and P. Manneville, J. Phys. (Paris) **40**, L609 (1979).
[16] See, for instance, D.W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations* (Clarendon, Oxford, 1987), Chap. 9.
[17] F. Haake, J.W. Haus, and R. Glauber, Phys. Rev. A **23**, 3255 (1981).
[18] For $D > 0$ the asymptotic configurations given by Eq. (1) are not strictly periodic, but have an almost constant number of nodes (see Refs. [19, 21]).
[19] E. Hernández-García, M. San Miguel, R. Toral, and J. Viñals, Physica (Amsterdam) **61D**, 159 (1992).
[20] L. Kramer, H.R. Schober, and W. Zimmermann, Physica (Amsterdam) **31D**, 212 (1988).
[21] J. Viñals, E. Hernández-García, M. San Miguel, and R. Toral, Phys. Rev. A **44**, 1123 (1991).
[22] P. Colet, F. de Pasquale, and M. San Miguel, Phys. Rev. A **43**, 5296 (1991).