Langevin Equations for Squeezing by Means of Non-linear Optical Devices

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Abstract. A procedure which enables one to obtain Langevin equations from the Heisenberg equations of motion representative of a wide class of non-linear optical devices is applied to the case of two-mode squeezing in a parametric device where allowance is made for different damping constants for the signal and idler modes, and a degenerate parametric amplifier with a fourth-order (Kerr effect) interaction term.

1 Introduction

The customary procedure for the derivation of Langevin equations for multimode optical fields generated by non-linear optical devices involves several steps [1]. First, a master equation for the motion of a density matrix has to be constructed from an effective Hamiltonian which includes the free-field, interaction and damping terms, after adiabatic elimination of the fast (heat-bath) modes, for which some heuristic assumptions regarding time-scale separation are usually considered. A generalized Fokker-Planck equation for an adequate pseudoprobability density is obtained next, after expansion of the density matrix on a suitable basis set. From standard representation theory [2] it is known that the different ways of ordering of the non-commuting operators lead to different equations of motion and that such orderings are equivalent provided that a correct representation exists. Different statistical properties will result as a consequence of the use of different representations since the equations of motion will evolve in time in an ordering-dependent way. However, since most of the observables are related to photo-counting devices, the normally ordered P (or generalized P) representations have been widely used [3], although in principle any other representation could also be employed if the final distribution is converted into normal order before the computation of field-statistical properties.

The debate regarding the physical soundness of the different representations is still open [4-6], and within it, a recent paper has emphasized the usefulness of the Wigner representation since it allows a direct derivation of the Langevin equations for the optical fields involved. Such a procedure is exact, provided that the effective Hamiltonian contains terms up to second order in the field operators. For higher-order problems, it is well known that the resulting Fokker-Planck equation for the Wigner quasi-probability distribution will, in general contain derivatives higher than second order, and therefore a truncation of those (or a linearization of the operator evolution equations) becomes necessary for obtaining a tractable solution. The available evidence suggests that [4,5] for a certain class of problems such as those related to quantum tunneling between the two stable states of a parametric oscillator operating above threshold, the full quantum problem should be considered. However, for problems not aiming to study the detailed dynamics near threshold and in circumstances where the noise is low in comparison with the mean values of the field amplitudes, the procedure described in [6] seems appropriate. In what follows we will briefly describe the application of the approach given in [6] to two representative cases of non-linear devices currently used for the generation of squeezed light.

2 A non-degenerate parametric amplifier with non-homogeneous linear losses

Apart from its interest as a squeezing device, some recent interest has arisen since it has been suggested that such a setup could be employed for high-accuracy (below shot-noise) absorption measurements [7]. The model herein considered represents an intense (assumed undepleted) laser beam of frequency 2Ω, which upon illumination of a suitable non-linear medium characterized by a second order coupling produces pairs of highly correlated photons with frequencies $\Omega \pm \epsilon$ where $\epsilon < \Omega$ is the modulation frequency.

The effective Hamiltonian written in the Schrödinger picture is written in terms of free-field, interaction and dissipative parts as

$$H = H_0 + H_{int} + H_D \tag{1}$$

$$H_0 = \omega_+ a_+^\dagger a_+ + \omega_- a_-^\dagger a_-$$

$$H_{int} = \frac{1}{2} f(t) a_+^\dagger a_-^\dagger + \frac{1}{2} f^*(t) a_+ a_-$$

$$H_D = \sum_j \omega_j b_j^+ b_j + \sum_j \chi_j^+ (b_j a_+^+ + b_j^- a_+) + \sum_j \chi_j^- (b_j a_-^+ + b_j^+ a_-)$$

where the function $f(t)$ contains the coupling terms $f(t) = i\kappa(t)e^{2i(\phi - \Omega t)}$ and the absorption losses have been included in the usual way in terms of the b_j heat–bath operators.

The Heisenberg equations of motion for the a_+, a_- and b_j modes can be written with ease and the heat–bath modes are then adiabatically eliminated.

By means of the correspondence $a_\pm \longrightarrow \alpha_\pm$ (and $a_\pm^\dagger \longrightarrow \alpha_\pm^*$) the Langevin equations of motion can be obtained for the two complex quantities describing the output modes with the result

$$\dot{\alpha}_+ = -i(\Omega + \epsilon - i\gamma_+)\alpha_+ - if(t)\alpha_-^* + L_+(t)$$

$$\dot{\alpha}_- = -i(\Omega - \epsilon - i\gamma_-)\alpha_- - if(t)\alpha_+^* + L_-(t) \tag{2}$$

where the damping constants $\gamma_\pm$ denote the linear losses for the two modes and the terms $L_\pm(t)$ represent complex white noise with a mean variance given by

$$< L_\pm(t) >= 0 =< L_\pm(t)L_\pm(t') >$$

$$< L_\pm(t)L_\pm^*(t') >= 2D_\pm\delta(t - t') \tag{3}$$

in terms of diffusion constants $D_\pm$, which basically represent the spectral densities at the frequencies of the two modes of the heat–bath,

$$2D_\pm = \gamma_\pm(1 + 2\bar{n}) \tag{4}$$

where $\bar{n}$ is an effective temperature.

The solution of (2) is a Wigner probability density, and a Fokker–Planck equation can be derived after separating the two real parts of the complex stochastic processes

$$\alpha_\pm(t) = [x_\pm(t) + iy_\pm(t)]e^{i[(\Omega \pm \epsilon)t - \phi]} \tag{5}$$

which transforms the system (2) into

$$\dot{x}_\pm = -\gamma_\pm x_\pm + \kappa x_\mp + L_{\pm x}$$

$$\dot{y}_\pm = -\gamma_\pm y_\pm - \kappa y_\mp + L_{\pm y} \tag{6}$$

where

$$< L_{\pm x,y}(t) >= 0$$

$$< L_{\pm i}(t) L_{\pm j}(t') >= D_{\pm} \delta_{ij} \delta(t - t'), \quad i,j = x, y \ .$$

It is straightforward to obtain from (6) a Fokker–Planck equation for the Wigner function which contains derivatives up to second order. On the other hand, a stationary solution for (6) exists which, above threshold, can be written as

$$x_+(t) = [A_0 + \alpha(t)] e^{\lambda_+ t} + [B_0 + \beta(t)] e^{-\lambda_- t}$$

$$x_-(t) = \mu_1 [A_0 + \alpha(t)] e^{-\lambda_+ t} + \mu_2 [B_0 + \beta(t)] e^{-\lambda_- t} \tag{7}$$

where

$$\lambda_\pm = \frac{\gamma_+ + \gamma_-}{2} \pm \left[\left(\frac{\gamma_+ + \gamma_-}{2} \right)^2 - (\gamma_+ \gamma_- - \kappa^2) \right]^{1/2} ,$$

$$\mu_1 = \frac{\gamma_+ - \lambda_+}{\kappa}, \quad \mu_2 = \frac{\gamma_+ - \lambda_-}{\kappa}$$

$$\alpha(t) = \int_0^t dt' \frac{\mu_2 L_+(t') - L_-(t')}{\mu_2 - \mu_1} e^{\lambda_+ t'}; \ \beta(t) = \int_0^t dt' \frac{\mu_1 L_+(t) - L_-(t')}{\mu_1 - \mu_2} e^{\lambda_- t'}$$

and a solution for the $y_\pm$ variable can be easily found upon the corresponding changes of variable.

The second and fourth–order expectation values can now be computed and the detailed expression for second and fourth–order correlations will be given in a forthcoming paper [8].

The quantity of interest for absorption mesurements would be the spectrum of fluctuations of the intensity difference between the signal I_+ and idler I_- modes

$$S_D(\omega) = \int d\tau e^{-i\omega t} < I_+(\tau) - I_-(\tau), I_+(0) - I_-(0) > \tag{8}$$

which can be written in terms of the intra- and interbeam fluctuations as

$$S_D(\omega) = S_{++}(\omega) + S_{--}(\omega) - S_{+-}(\omega) - S_{-+}(\omega) \tag{9}$$

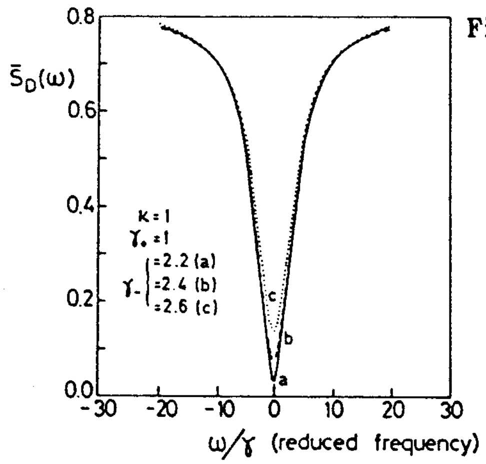

Figure 1

$$S_{++}(\omega)+S_{--}(\omega) = \int d\tau e^{-i\omega\tau}[< I_+(t+\tau), I_+(t) > + < I_-(t+\tau), I_-(t) >]$$

$$S_{+-}(\omega)+S_{-+}(\omega) = \int d\tau e^{-i\omega\tau}[< I_+(t+\tau), I_-(t) > + < I_-(t+\tau), I_+(t) >].$$

The evaluation of the spectral components can be obtained after some calculus and the detailed expressions will be given elsewhere [8].

A normalized spectrum may be defined as

$$\bar{S}_D(\omega) = \frac{S_D(\omega)}{S_{++}(\omega) + S_{--}(\omega)} \tag{10}$$

so that the shot-noise limit is $\bar{S}_D(\omega) = 1$ and perfect noise suppression means $S_D(\omega) = 0$. The shape of the spectrum for several values of the asymmetric losses is shown in Figure 1. As can be seen upon inspection of the figure, the increase in the loss coefficient of one of the modes leads to an increase in the intensity fluctuations and therefore it may be used in order to perform absorption measurements well below the shot-noise limit. A full detailed discussion will be given elsewhere [8].

3 Squeezing from fourth-order interaction in a degenerate parametric amplifier

It has been shown that [9] an intense coherent beam which propagates along an optical Kerr medium, exhibits "self-squeezing" (amplitude squeezing), and that the effect could be remarkably high provided that the incident pump strength is strong enough. The effect of such higher-order interactions coupled to a standard two-photon medium was analyzed by Tombesi [10] some time ago. The interest in the already mentioned study was focused on the possibilities of generating strong squeezed light at short interaction times (i.e. in travelling-wave geometries). Our interest here is to explore the field-statistical properties of a more realistic model of a parametric amplifier where the crystal losses are explicitly accounted for.

The effective Hamiltonian is written as

$$H = \Omega a^+ a + \frac{1}{2} f(t) a^{+^2} + \frac{1}{2} f^*(t) a^2 + \frac{\Gamma}{2} a^{+^2} a^2 + \sum_j \omega_j b_j^+ b_j + \\ + \sum_j \chi_j (b_j a^+ + b_j^+ a) \,, \qquad (11)$$

where the first term on the right-hand side is the free-field contribution, the second-order interaction is described by the terms which contain the function $f(t) = i\kappa(t)e^{2i(\phi-\Omega t)}$, where $\kappa(t)$ characterizes the coupling, and the fourth-order term is written in terms of the non-linear coupling coefficient Γ which is taken to be proportional to the third-order susceptibility. The interaction with the heat-bath modes is comprised in the last term.

We will follow an analogous procedure to the one previously used except that a linearization has to be introduced in order to deal with the higher-order term.

The Heisenberg equations for the field operators are readily obtained and upon the elimination of the heat-bath modes one gets

$$\dot{a} = -i(\Omega - i\gamma)a - if(t)a^* - i\Gamma a^* a^+ + L(t) \qquad (12)$$

$$< L(t)L^*(t) >= 2D\delta(t-t'), \quad 2D = \gamma(1+2\bar{n})$$

where γ is the damping constant and $L(t)$ are (complex) white noise terms. The linearization of the Heisenberg equation can be done provided that a separation can be made between the c-number part and the part carrying the quantum fluctuactions δa, that for the destruction operator becomes

$$a = (\bar{a} + \delta a)e^{-2i(\Omega t - \phi)} \,.$$

We can now replace the quantum noise term δa by the complex stochastic process $\delta\alpha$

$$\delta\dot{\alpha} = -i(\Omega - i\gamma)\delta\alpha - if(t)\delta\alpha^* - i\Gamma\bar{\alpha}(2\bar{\alpha}^*\delta\alpha + \bar{\alpha}\delta\alpha^*) + L(t) \qquad (13)$$

which has a solution given as a Wigner probability density. The associated Fokker–Planck equation in the W representation will thus correspond to a reduced form where the derivatives higher than second order have been truncated.

The complex $\alpha(t)$ stochastic processes can be converted into real ones as was done previously (Eqn. (5)) , and the Langevin equations for the two field quadratures will be

$$\dot{x} = (-\gamma + \kappa)x + \Gamma y(y^2 + x^2) + L_x$$

$$\dot{y} = (-\gamma - \kappa)y - \Gamma x(y^2 + x^2) + L_y$$

where the noise terms have the same properties as those given in (6).

By introduction of a rescaled time $\tau = t(\gamma + \kappa)$ and defining the dimensionless constants

$$\mu = \frac{\kappa - \gamma}{\kappa + \gamma}, \quad \nu = \frac{\Gamma}{\gamma + \kappa}$$

the equations can be written as

$$\dot{x} = \mu x + \nu y(x^2 + y^2) + L_x$$

$$\dot{y} = -y - \nu x(x^2 + y^2) + L_y \qquad (14)$$

which have a deterministic stationary solution which verifies

$$\frac{\bar{x}_s^2}{\bar{y}_s^2} = \frac{1}{\mu} \ .$$

In consequence, it can easily be verified that for $\mu < 0$ the trivial solution $(x_s, y_s) = (0,0)$ is obtained and two other stationary solutions are found above threshold (i.e. for $\mu > 0$)

$$(\bar{x}_s, \bar{y}_s) = (0,0) \quad \text{for } \mu < 0$$

$$\bar{x}_s = \mp[\mu^{1/2}/\nu(1+\mu)]^{1/2}; \quad \bar{y}_s = -\mu^{1/2}\bar{x}_s \ \text{ for } \mu > 0 \qquad (15)$$

A local stability analysis can be carried out by means of a linearization around the deterministic solution and one finds that below threshold only the trivial solution is stable whereas a saddle point is approached at threshold and the two solutions above threshold become the only stable ones (a 'pitchfork'-type bifurcation is developed as one passes through the threshold).

In the limit of vanishing damping ($\nu = 1$) the system becomes of Hamiltonian nature and evolves according to

$$\dot{x} = x + \nu y(y^2 + x^2) = \frac{\partial H(x,y)}{\partial y}$$

$$\dot{y} = -y - \nu x(y^2 + x^2) = -\frac{\partial H(x,y)}{\partial x}, \quad H(x,y) = xy + \frac{\nu}{4}(x^2 + y^2)^2 .$$

From the equations (14) the fluctuations around the steady states as well as the effects due to the presence of the non-linear term can be computed and some results are presented in an accompanying paper [11].

4 Summary

The method of computing the statistical properties of non-linear optical devices described in [6] which allows one to obtain the field-statistical properties from the Heisenberg equations of motion has been applied to two representative classes of model Hamiltonians. In the first case, since the model involved only contains terms up to second order, the obtained results should be considered as exact as the ones obtained by means of the standard P representation. Since a linearization has been introduced on the second case, the results have a range of validity which may exclude the parameter region located near threshold. A more detailed discussion on this topic is contained in an accompanying paper [11].

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