Synchronization of Chaotic Systems by Common Random Forcing

Raúl Toral[1,2], Claudio R. Mirasso[2], E. Hernández-García[1,2] and Oreste Piro[1,2]

(1)Instituto Mediterráneo de Estudios Avanzados, IMEDEA[1], CSIC-UIB.
(2) Departament de Física, Universitat de les Illes Balears.
07071-Palma de Mallorca, Spain

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Abstract. We show two examples of noise–induced synchronization. We study a 1-d map and the Lorenz systems, both in the chaotic region. For each system we give numerical evidence that the addition of a (common) random noise, of large enough intensity, to different trajectories which start from different initial conditions, leads eventually to the perfect synchronization of the trajectories. The largest Lyapunov exponent becomes negative due to the presence of the noise terms.

I INTRODUCTION

The issue of whether chaotic systems can be synchronized by common random noise sources has attracted much attention recently [1–6]. It has been reported that for some chaotic maps, the introduction of the same (additive) noise in independent copies of the same map could lead (for large enough noise intensity) to a collapse onto the same trajectory, independently of the initial condition assigned to each of the copies [1]. This synchronization of chaotic systems by the addition of random terms is a remarkable and counterintuitive effect of noise. Nowadays, some contradictory results exist for the existence of this phenomenon of noise–induced synchronization. It is purpose of this paper to give explicit examples in which it is shown that one can indeed obtain such a synchronization. Moreover, the examples open the possibility to obtain such a synchronization in electronic circuits, hence suggesting that noise-induced synchronization of chaotic circuits can indeed be used for encryption purposes.

Although the issue of which is the effect of noise in chaotic systems was considered at the beginning of the 80's [7], to our knowledge the first atempt to synchronize

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two chaotic systems by using the same noise signal was considered by Maritan and Banavar [1]. These authors analysed the logistic map in the presence of noise:

$$x_{n+1} = 4x_n(1 - x_n) + \xi_n \tag{1}$$

where ξ_n is the noise term, considered to be uniformly distributed in a symmetric interval $[-W, +W]$. They showed that, if W was large enough (i.e. for a large noise intensity) two different trajectories which started with different initial conditions but used otherwise the same sequence of random numbers, would eventually coincide into the same trajectory. This result was heavily critisized by Pikovsky [8] who argued that two systems can synchronize only if the largest Lyapunov exponent is negative. He then shows that the largest Lyapunov exponent of the logistic map in the presence of noise is always positive and concludes that the synchronization is, in fact, a numerical effect of lack of precision of the calculation. Furthermore, Malescio [2] pointed out that the noise used to simulate Eq.(1) in [1] was not really symmetric. This is because the requirement $x_n \in (0, 1)$, $\forall n$, actually leads to discard those values for the random number ξ_n, which do not fulfill such condition. The average value of the random numbers which have been accepted is different from zero, hence producing an effective *biased* noise, i.e. one which does not have zero mean. The introduction of a non-zero mean noise means that we are altering essentially the properties of the deterministic map.

Noise induced synchronization has been since studied for other chaotic systems such as the Lorenz model [1,2] and the Chua circuit [3,5]. Synchronization of trajectories starting with different initial conditions but using otherwise the same sequence of random numbers was observed in the numerical integration of a Lorenz system in the presence of a noise distributed uniformly in the interval $[0, W_L]$, i.e. again a noise which does not have a mean of zero [1]. Other detailed studies [2] also conclude that it is not possible to synchronize trajectories in a Lorenz system by adding an unbiased noise. Similarly, the studies of the Chua circuit always conclude that a biased noise is needed for synchronization [5]. Therefore a widespread belief exists that it is not possible to synchronize two chaotic systems by injecting the same noisy signal to both of them. However, in this paper we give numerical evidence that it is possible to synchronize two chaotic systems by the addition of a common noise which is Gaussian distributed and not biased. We analyse specifically a 1-d map and the Lorenz system, both in the chaotic region. The necessary criterion introduced in ref. [8] is fully confirmed and some heuristic arguments are given about the general validity of our results. Finally, we conclude with some open questions relating the general validity of our results.

II RESULTS

The first example is that of the map:

$$x_{n+1} = f(x_n) + \epsilon\xi_n \tag{2}$$

where ξ_n is a set of uncorrelated Gaussian variables of zero mean and variance 1. We use explicitely

$$f(x) = \exp\left[-\left(\frac{x-0.5}{\omega}\right)^2\right] \tag{3}$$

We plot in Fig.(1a) the bifurcation diagram of this map. In the noiseless case, we can see the typical windows in which the system behaves chaotically. The associated Lyapunov exponent, λ, in these regions is positive. For instance, for $\omega = 0.3$ (the case we will be considereing throughout the paper) it is $\lambda \approx 0.53$. In Fig.(1b) we observe that the Lyapunov exponent becomes negative for most values of ω for large enough noise level ϵ. Again for $\omega = 0.3$ and now for $\epsilon = 0.2$ it is $\lambda = -0.17$. For the noiseless case, it is $\lambda > 0$ and trajectories starting with differential initial conditions, obviously, remain different for all the iteration steps, see Fig.(2a). However, when moderated levels of noise ($\epsilon \geq 0.1$) are used, λ becomes negative and trajectories starting with different initial conditions, but using the same sequence of random numbers, synchronize perfectly, see Fig.(2b).

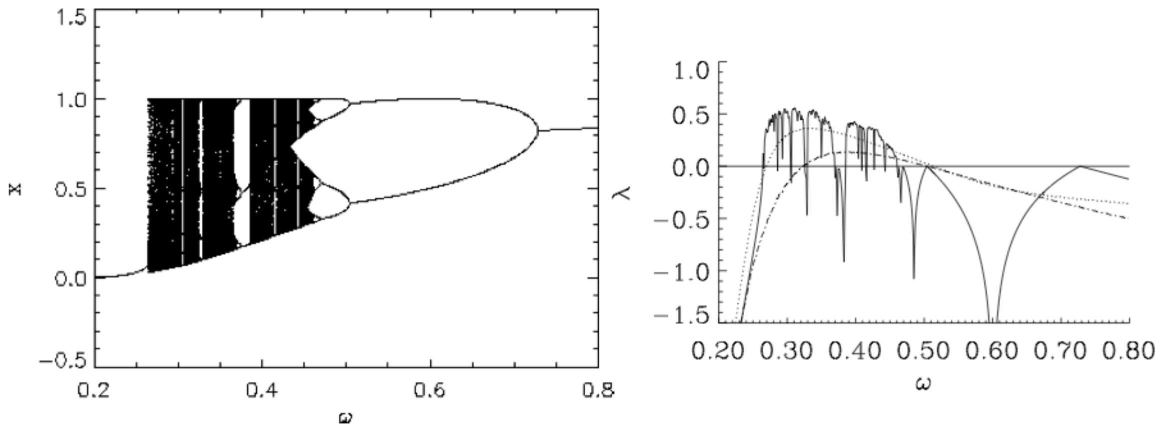

FIGURE 1. (a) Bifurcation diagram of the map given by Eqs.(2) and (3) in the absence of noise terms. (b) Lyapunov exponent for the noiseless map ($\epsilon = 0$, continuous line) and the map with a noise intensity $\epsilon = 0.1$ (dotted line) and $\epsilon = 0.2$ (dot-dashed line).

According to [8], convergence of trajectories to the same one, or loss of memory of the initial condition, can be stated as *negativity of the Lyapunov exponent*. The Lyapunov exponent of a map $x_{n+1} = F(x_n)$ is defined as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \ln |F'(x_i)| \tag{4}$$

It is the average of (the logarithm of the absolute value of) the successive slopes F' found by the trajectory. Slopes in $[-1, 1]$ contribute to λ with negative values, indicating trajectory convergence. Larger or smaller slopes contribute with positive values, indicating trajectory divergence. Since the deterministic and noisy maps

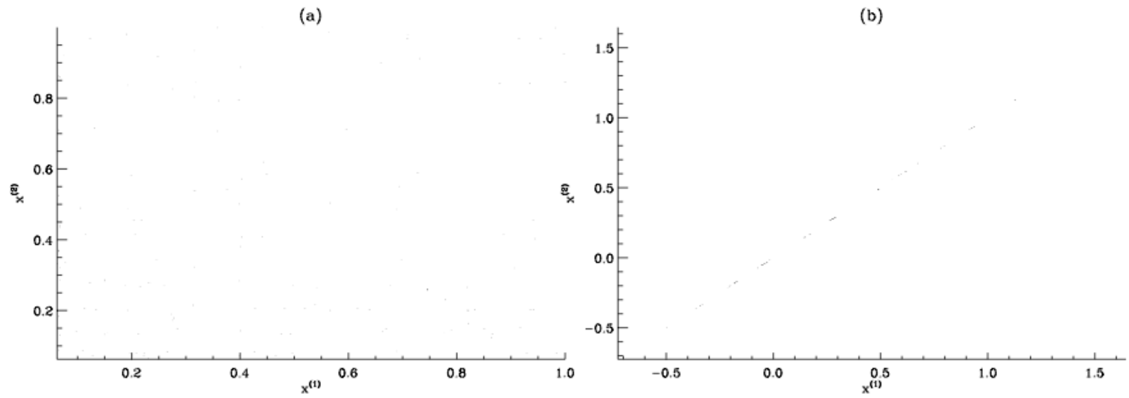

FIGURE 2. Plot of two realizations $x^{(1)}$, $x^{(2)}$ of of the map given by Eqs. (2) and (3). Each realization consists of 10,000 points which have been obtained by iteration of the map starting in each case from a different initial condition (100,000 initial iterations have been discarded and are not shown). In figure (a) there is no noise, $\epsilon = 0$ and the trajectories are independent of each other. In figure (b) we have use a level of noise $\epsilon = 0.2$ producing a perfect synchronization (after discarding some initial iterations).

satisfy $F' = f'$ one is tempted to conclude that the Lyapunov exponent is not modified by the presence of noise. However, there is noise-dependence through the trajectory values x_i, $i = 1, ..., N$. In the absence of noise, λ is positive, indicating trajectory separation. When synchronization is observed, the Lyapunov exponent becomes negative, as required by the argument in [8].

By using the definition of the *invariant measure on the attractor*, or *stationary probability distribution* $P_{st}(x)$, the Lyapunov exponent can be calculated also as

$$\lambda = \langle \log |F'(x)| \rangle = \langle \log |f'(x)| \rangle \equiv \int P_{st}(x) \log |f'(x)| dx \tag{5}$$

Here we see clearly the two contributions to the Lyapunov exponent: although the derivative $f'(x)$ does not change when including noise in the trajectory, the stationary probability does change (see Fig.3), thus producing the observed change in the Lyapunov exponents. Synchronization, then, can be a general feature in maps which have a large region in which the derivative $|f'(x)|$ is smaller than one. Noise will be able, then , to explore that region and yield, on the average, a negative Lyapunov exponent.

The second system we have studied is the well known Lorenz model with random terms added [9,1]:

$$\begin{aligned} \dot{x} &= p(y - x) \\ \dot{y} &= -xz + rx - y + \epsilon\xi \\ \dot{z} &= xy - bz \end{aligned} \tag{6}$$

ξ now is white noise: a Gaussian random process of mean zero and delta correlated, $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. We have used $p = 10$, $b = 8/3$ and $r = 28$ which, in the deterministic case, $\epsilon = 0$ are known to lead to a chaotic behavior (the largest Lyapunov

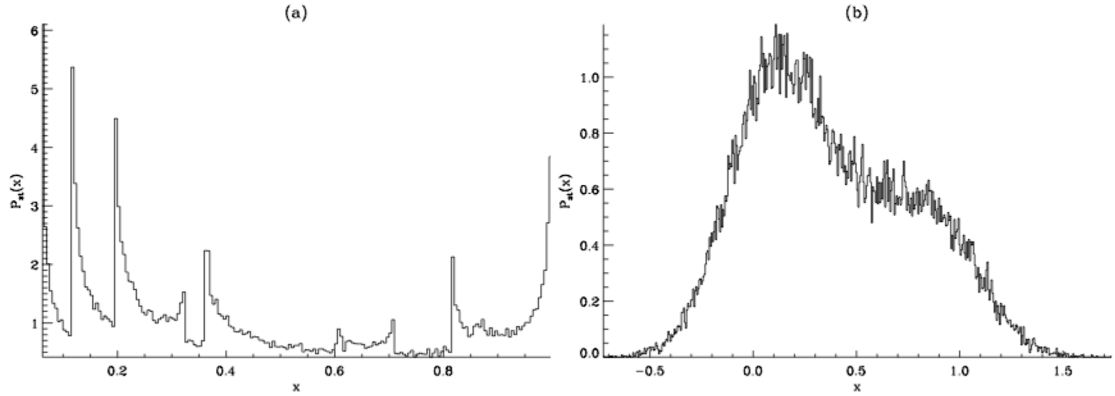

FIGURE 3. Plot of the stationary distribution for the map given by Eqs.(2) and (3) in the (a) deterministic case $\epsilon = 0$, and (b) the case with noise along the trajectory, $\epsilon = 0.2$.

exponent is $\lambda \approx 0.9 > 0$). We have integrated numerically the above equations using the Euler method with a time step $\Delta t = 0.001$. For the deterministic case, trajectories starting with different initial conditions are completely uncorrelated, see Fig. (4a). This is also the situations for small values of ϵ. However, when using a noise intensity $\epsilon = 40$ the noise is strong enough to induce synchronization of the trajectories. Again the presence of the noise terms makes the largest Lyapunov exponent become negative (for $\epsilon = 40$ it is $\lambda \approx -0.2$). As in the example of the map, after some transient time, two different evolutions which have started in completely different initial conditions synchronize towards the same value of the three variables (see Fig. (4b) for the z coordinate). One could argue that the intensity of the noise is very large. However, the basic structure of the "butterfly" Lorenz attractor remains unchanged as shown in Fig. (5).

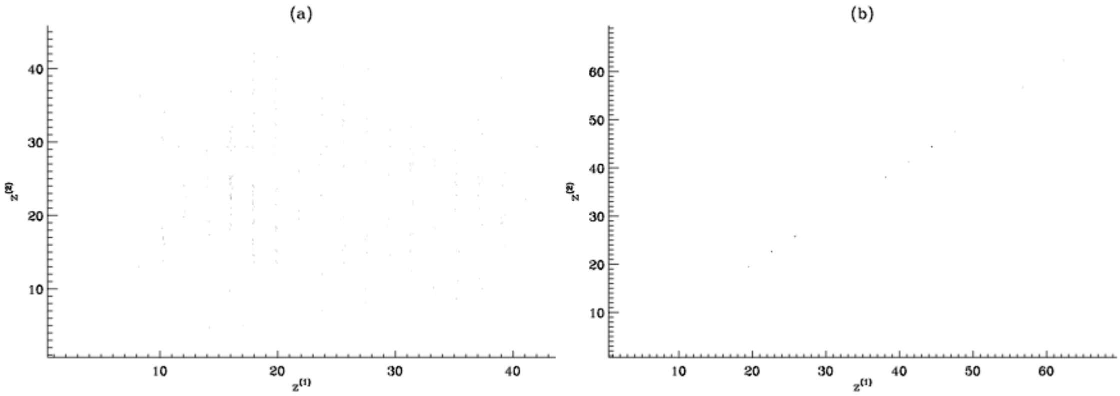

FIGURE 4. Same than figure (1) for the z variable of the Lorenz system, Eqs.(6) in the (a) deterministic case $\epsilon = 0$ and (b) $\epsilon = 40$. Notice the perfect synchronization in case (b).

In conclusion, we have shown that it is possible for noise to synchronize trajectories of a system which, deterministically, is chaotic. The novelty of our results is that the noise used in the two examples, a 1-d map and the Lorenz system, is unbiased, i.e. has always zero mean.

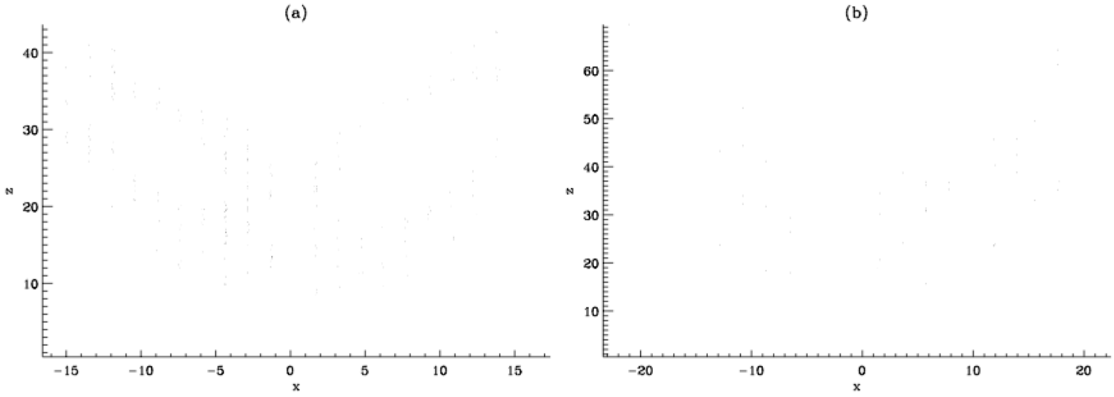

FIGURE 5. "Butterfly" attractor of the Lorenz system in the cases (a) of no noise $\epsilon = 0$ and (b) $\epsilon = 40$.

There still remain many open questions in this field. They involve the development of a general theory, probably based in the invariant measure, that could give us a criterion to determine the range of parameters (including noise levels) for which the Lyapunov exponent becomes negative, thus allowing synchronization. Another important question relates the structural stability of this phenomenon. Any practical realization of this system can not produce two *identical* samples. If one wants to use sthocastic synchronization of electronic emitters and receivers (as a means of encryption) one should be able to determine which is the allowed discrepancy between circuits before the lack of synchronization becomes unacceptable.

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